

The Upper Total x -Edge Steiner Number of a Graph

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Let x be a vertex of a connected graph G and $W \subseteq V(G)$. Then W is called a total x -edge Steiner set of G if W is an x -edge Steiner set of G and $\langle W \rangle$ has no isolated vertices. The minimum cardinality of a total x -edge Steiner set of G is defined as total x -edge Steiner number of G and denoted by $s_{t1x}(G)$. Any total x -edge Steiner set of cardinalities $s_{t1x}(G)$ is called a s_{t1x} -set of G . A total x -edge Steiner set W in a connected graph G is called a minimal total x -edge Steiner set if no proper subset of W is a total x -edge Steiner set of G . The upper total x -edge Steiner number $s_{t1x}^+(G)$ of G is the maximum cardinality of a minimal total x -edge Steiner set of G . The upper total x -edge Steiner number of certain classes of graph are determined. Some general properties satisfied by this concept are studied. For any positive integers a and b with $4 \leq a \leq b$, there exists a connected graph G such that $s_{t1x}(G) = a$ and $s_{t1x}^+(G) = b$.

Keywords: Edge Steiner Number, x -edge Steiner Number and Upper Total x -edge number.

Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. The distance $d(u, v)$ between two vertices u and v in a connected graph

G is the length of a shortest $u-v$ path in G . An $u-v$ path of length $d(u, v)$ is called an $u-v$ geodesic. For basic graph theoretic terminology, we refer to Harary [1].

For a nonempty set W of vertices in a connected graph G , the Steiner distance $d(W)$ of W is the minimum size of a connected subgraph of G containing W . Necessarily, each such subgraph is a tree and is called a Steiner tree with respect to W or a Steiner W -tree. It is to be noted that $d(W) = d(u, v)$, when $W = \{u, v\}$. The set of all vertices of G that lie on some Steiner W -tree is denoted by $S(W)$. If $S(W) = V$, then W is called a Steiner set for G . A Steiner set of minimum cardinalities is a minimum Steiner set or simply a s -set of G and this cardinality is the Steiner number $is(G)$ of G . The Steiner number of a graph was introduced and studied in [2] and further

studied in [3,4,5,6]. Let x be a vertex of a connected graph G and $W \subset V(G)$. Then W is called a total x -edge Steiner set of G if W is an x -edge Steiner set of G and $\langle W \rangle$ has no isolated vertices. The minimum cardinality of a total x -edge Steiner set of G is defined as total x -edge Steiner number of G and denoted by $s_{t1x}(G)$. Any total x -edge Steiner set of cardinalities $s_{t1x}(G)$ is called a s_{t1x} -set of G . This concept was studied in [10].

Theorem 1.1. [10] Every extreme vertex of G other than the vertex x (whether x is extreme or not) belongs to every total x -edge Steiner set for any vertex x in G .

Theorem 1.2. [10] For the complete graph $G = K_p$ ($p > 2$), $s_{t1x}(K_p) = p - 1$ for every vertex x .

The Upper Total x -Edge Steiner Number of a Graph

Definition 2.1. A total x -edge Steiner set W in a connected graph G is called a minimal total x -edge Steiner set if no proper subset of W is a total x -edge Steiner set of G . The upper total x -edge Steiner number $s_{t1x^+}(G)$ of G is the maximum cardinality of a minimal total x -edge Steiner set of G .

Example 2.2. For the graph G in Figure 2.1, the minimum total x -edge Steiner sets and the minimal total x -edge Steiner numbers are given in the following table.

Table 2.1

vertex x	s_{t1x} -sets	minimal total x -edge Steiner set	$s_{t1x^+}(G)$
v_1	$\{v_4, v_5\}$	$\{v_3, v_4, v_5, v_6\}$	4
v_2	$\{v_1, v_2, v_4, v_5\}$	$\{v_1, v_2, v_4, v_5\}$	4
v_3	$\{v_1, v_2, v_4, v_5, v_6\}$	$\{v_1, v_2, v_4, v_5, v_6\}$	5
v_4	$\{v_1, v_2, v_4, v_5\}$	$\{v_1, v_2, v_4, v_5\}$	4
v_5	$\{v_1, v_2, v_4, v_5\}$	$\{v_1, v_2, v_4, v_5\}$	4
v_6	$\{v_1, v_2, v_3, v_4, v_5\}$	$\{v_1, v_2, v_3, v_4, v_5\}$	5

Theorem 2.3. Each extreme vertex of a graph G belongs to minimal total x -edge Steiner set of G . In particular, each end-vertex of G belongs to every minimal total x -edge Steiner set of G .

Proof. Since every minimal total x -edge Steiner set of G is a x -edge Steiner set of G . Hence follows from Theorem 1.1.

Theorem 2.4. For a connected graph G , $2 \leq s_{t1x}(G) \leq s_{t1x^+}(G) \leq p$.

Proof. Since any total x -edge Steiner set needs at least two vertices, $s_{t1x}(G) > 2$. Let W be a minimum total x -edge Steiner set of G so that $s_{t1x}(G) = |W|$. Since W is also minimal total x -edge Steiner set of G , it is clear that $s_{t1x}(G) \leq s_{t1x^+}(G) = |W|$. Hence $s_{t1x}(G) \leq s_{t1x^+}(G)$. Since the vertex set $V(G)$ is a total x -edge Steiner set, $s_{t1x^+}(G) \leq p$. Thus $2 \leq s_{t1x}(G) \leq s_{t1x^+}(G) \leq p$.

Remark 2.5. The bound in Theorem 2.4 is sharp. For the graph G in Figure 2.2. Let $x = v_4$ is a total x -edge Steiner set of G so that $s_{t1x}(G) = 2$. For the non-trivial tree T , is not a star, $s_{t1x}(G) = \lfloor s \rfloor_{t1x^+}(G)$ for any internal vertex T . For the star, $K_{(1, p-1)}$, $s_{t1x^+}(G) = p$ for the internal vertex x .

Theorem 2.6. For a connected graph G , $s_{t1x}(G) = p$ if and only if $s_{t1x^+}(G) = p$ for some vertex x in G .

Proof. Let $s_{t1x}(G) = p$. Then $S = V(G)$ is the unique minimal total x -edge Steiner set of G . Since no proper subset of S is a total x -edge Steiner set, it is clear that S is the unique minimum total x -edge Steiner set of G and so $s_{t1x}(G) = p$. The converse follows from Theorem 2.4.

Theorem 2.7. For the complete graph K_p ($p > 2$), $s_{t1x^+}(K_p) = p-1$ for every vertex x in G .

Proof. Since every vertex of the complete graph K_p ($p \geq 2$) is an extreme vertex, the vertex set of K_p is the unique total x -edge Steiner set of K_p . Thus $s_{t1x}(K_p) = p - 1$. Hence by Theorem 1.2 $s_{t1x^+}(K_p) = p - 1$.

Theorem 2.8. If G is a connected graph of order p with $s_{t1x}(G) = p - 1$, then $s_{t1x^+}(G) = p - 1$ for every vertex x in G .

Proof. Since $s_{t1x}(G) = p-1$, it follows from Theorem 2.4 that $s_{t1x}(G) = p$ or $p-1$. If $s_{t1x}(G) = p$, then by Theorem 2.6, $s_{t1x}(G) = p$, which is a contradiction. Hence $s_{t1x^+}(G) = p - 1$.

Corollary 2.9. For a star, $G = K_{(1,p-1)}$, $\lfloor s_{t1x}(G) = s \rfloor_{t1x^+}(T) = \{(p \text{ if for every internal vertex } x \text{ in } G @ p-1 \text{ if every end vertex of } G) \}$

Theorem 2.10. For a tree T , $s_{t1x}(T) = s_{t1x^+}(T) = k + 1$, where k is the number of end vertices and l is the number of support vertices.

Proof. By Theorems 2.3, $s_{t1x}(T) = s_{t1x^+}(T) = k+1$ for all x in T . Let Z be the set of all end vertices of T and S be the set of all support vertices of T . Then $W = SZ$ is the unique minimal total x -edge Steiner set of T and that $s_{t1x}(T) = s_{t1x^+}(T) = k + 1$.

Theorem 2.11. For a complete bipartite graph $G = K_{m,n}$, ($2 \leq m \leq n$), $s_{t1x^+}(G) = m+n-1$ for some vertex x in G .

Proof. Let $U = \{u_1, u_2, \dots, u_m\}$, and $V = \{v_1, v_2, \dots, v_n\}$ be the bipartition of G . Let $x \in G$. Then $S = G - \{x\}$ is the unique minimal total x -edge Steiner set of G so that $s_{t1x^+}(G) = |W| = m+n-1$.

Theorem 2.12. For every positive integer a and b with $4 \leq a \leq b$, there exists a connected graph G such that $s_{t1x}(G) = a$ and $s_{t1x^+}(G) = b$.

Proof. If $a = b$, take $G = K_{(1, a-1)}$. Then by Theorem 2.10, $s_{t1x}(G) = s_{t1x^+}(G) = a$ for the internal vertex x . So, let $a < b$. Let G be the graph by obtained in Figure 2.3 from the path on five vertices $P: v_1, v_2, v_3, v_4, v_5$ by adding the new vertices $w_1, w_2, \dots, w_{(b-a+1)}$ and $z_1, z_2, \dots, z_{(a-4)}$ and joining each w_i ($1 \leq i \leq b - a + 1$) with v_1 and v_4 and also joining each z_i ($1 \leq i \leq a - 4$) with v_4 . Let $x = v_2$. Let $W = \{z_1, z_2, \dots, z_{(a-4)}, v_2, v_4, v_5\}$ be the set of end vertices and cut vertices of G . Let S be any total x -edge Steiner set of G . Then by Theorems 2.3, $W \subseteq S$. Then W is not a total x -edge Steiner set of G and so $s_{t1x}(G) \geq a$. However, $W \cup \{v_1\}$ is a total x -edge Steiner set of G so that $s_{t1x}(G) = a$.

Now, $T = W \cup \{w_1, w_2, \dots, w_{(b-a+1)}\}$ is a total x -edge Steiner set of G . We show that T is a minimal total x -edge Steiner set of G . Let W be any proper subset of T . Then there exist at least are vertex say $u \in T$ such that $u \notin W$. By Theorem 1.1, $u \neq z_i$ ($1 \leq i \leq a - 4$). Also, by Theorem 1.1 $u \neq v_2, v_4$. Now, assume that $u = v_j$ for some j ($1 \leq j \leq b - a + 1$). Then the vertex v_j ($1 \leq j \leq b-a+1$) does not lie on any Steiner W -tree of G . Now, assume that $u = v_1$. Then the vertex v_1 does not lie on any Steiner W -tree of G . Now assume that $u = v_3$. Then the vertex v_3 does not lie on any Steiner W -tree of G and so W is not a total x -edge Steiner set of G . Hence T is a minimal total x -edge Steiner set of G so that $s_{t1x^+}(G) = b$. Now, we show that there are no minimal total x -edge Steiner set X of G with $|X| \geq b + 1$. Since $|V(G)| = b + 2$, there is no total x -edge Steiner set with $|X| = b+1$. Thus $s_{t1x^+}(G) = b$.

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